DEFORMABLE MODEL USING RADIAL BASIS FUNCTIONS BASED LEVEL SET INTERPOLATION WITH AN ELLIPSE CONSTRAINT

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ABSTRACT

A level-set-based method using a radial basis functions (RBFs) based level set interpolation with an ellipse constraint is presented for image contour extraction. In the present method, the initial distance function embedded in the ellipse-constrained RBFs is interpolated using a coarse grid. The deformation of the level set function (LSF) is considered as an update of the RBFs' coefficients by solving an ordinary differential equation (ODE) and non-convex constrained quadratic programming (QCQP). A semi-definite relaxation approach is proposed to solve the non-convex QCQP problem. The proposed level set evolving scheme, which does not need initialization and re-initialization, is efficient and does not suffer from self-flattening. The objects with extremely complex shapes can be exactly fitted with a coarse grid of RBFs' centers and the image extraction is less sensitive to the distribution of the objects in the image domain.

Keywords: Image Segmentation, Level Set Interpolation, Radial Basis Functions, Deformable Model, Constrained Quadratic Programming

1. INTRODUCTION

The level-set-based methods reported in the literature belong to a class of deformable models for segmenting images with complex topology¹. In the conventional level-set-based methods, the zero level set of a smooth function (usually called the level-set function, LSF) propagates in an evolution process to capture an unknown number of objects. The curves generated from LSFs are not required to be re-parameterized and the evolving contours split and merge naturally, allowing a simultaneous detection of both the interior and exterior boundaries of multiple objects. However, the original level-set-based deformable model has limited use in the real applications as it suffers from weak edge leakage². Moreover, it is not uncommon that the LSF develops irregularities during its evolution, which can cause numerical errors and eventually destroy the stability of the evolution³. A great deal of research has been carried out to overcome the shortcomings of the original level-set method^{4, 5, 6, 7, 8, 9}. For instance, Chan and Vese⁴ considered the deformable model as an energy minimization of a Mumford–Shah functional⁵. Wang et al.⁶ interpolated LSFs using radial basis functions (RBFs) and transformed the Hamilton–Jacobi partial differential equations (PDEs) into a system of ODEs for structural topology optimization. In this paper, the LSF is expressed as RBFs with an ellipse constraint. A novel method for the evolution of the RBFs based level-set with an ellipse constraint is also proposed. Several examples are given to illustrate the applications of the proposed method.

2. PROPOSED METHOD

2.1 Theoretical Review

2.1.1 Level-set representation

In the level-set formalism¹⁰, the contour/surface is implicitly represented by a multi-dimensional scalar function with the moving front embedded at the zero level set. Let $C \in R^2$ denote the moving front, the level-set function ϕ must satisfy

$$\begin{cases} \phi(\mathbf{x}, t) > 0, & \mathbf{x} \in \Omega_{in}(t) \\ \phi(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega_{in}(t) = C(t) \\ \phi(\mathbf{x}, t) < 0, & \mathbf{x} \in \Omega_{out}(t) \end{cases} \quad (1)$$

where Ω_{in} is a region in the image domain Ω bounded by C ; $\mathbf{x}$ is a point $(x, y) \in \Omega$. Ω_{out} is defined as $\Omega_{out} = \Omega - \Omega_{in}$.

2.1.2 Chan-Vese deformable model

In the Chan-Vese model, the deformable contour represented implicitly by the level-set function evolves to capture all the object boundaries by minimizing a functional

$$J^c(\mu_{in}, \mu_{out}, \phi) = \int_{\Omega} (I(\mathbf{x}) - \mu_{in})^2 H(\phi(\mathbf{x})) d\mathbf{x} + \int_{\Omega} (I(\mathbf{x}) - \mu_{out})^2 (1 - H(\phi(\mathbf{x}))) d\mathbf{x} + \nu \int_{\Omega} \delta(\phi(\mathbf{x})) \|\nabla \phi(\mathbf{x})\| d\mathbf{x} \quad (2)$$

where $I(\mathbf{x})$ is the intensity at the point $\mathbf{x}$, $\phi(\mathbf{x})$ is the level-set value at the pixel $\mathbf{x}$, and ν is the regularization parameter controlling the degree of the smoothness of the segmentation. μ_{in} , μ_{out} are average intensities across the inside region (Ω_{in}) and outside region ($\Omega_{out} = \Omega - \Omega_{in}$), respectively. Herein, μ_{in} and μ_{out} are computed at each iteration of the evolution process using the following expressions:

$$\begin{aligned} \mu_{in} &= \frac{\int_{\Omega} I(\mathbf{x}) H(\phi(\mathbf{x})) dxdy}{\int_{\Omega} H(\phi(\mathbf{x})) dxdy} \\ \mu_{out} &= \frac{\int_{\Omega} I(\mathbf{x}) (1 - H(\phi(\mathbf{x}))) dxdy}{\int_{\Omega} (1 - H(\phi(\mathbf{x}))) dxdy} \end{aligned} \quad (3)$$

where $H(z)$ is the Heaviside function. As explained in Luo et al.¹¹, the Heaviside function can be evaluated using its smeared-out version $H_{\varepsilon}(\phi)$ with $\varepsilon = 1.5$ pixel, i.e.,

$$H_{\varepsilon}(\phi) = \frac{1}{2} \left(1 + \frac{2}{\pi} \arctan\left(\frac{\phi}{\varepsilon}\right) \right) \quad (4)$$

Keeping μ_{in} , μ_{out} fixed, $J^c(\mu_{in}, \mu_{out}, \phi)$ is minimized with respect to ϕ using the standard gradient descent method by solving the gradient flow equation as follows:

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = F(\mathbf{x}, t) |\nabla \phi(\mathbf{x}, t)| \quad (5)$$

where F is the speed function given as

$$F(\mathbf{x}, t) = (I(\mathbf{x}) - \mu_{in})^2 - (I(\mathbf{x}) - \mu_{out})^2 + v \operatorname{div} \left(\frac{\nabla \phi(\mathbf{x}, t)}{|\nabla \phi(\mathbf{x}, t)|} \right)$$

This algorithm is a region-based method, which tends to separate the image into two homogeneous regions (according to their mean value). ϕ is a signed distance function and is reinitialized at each iteration. The evolution is only computed on the narrow-band of the level-set thus making it sensitive to initialization.

2.2 Radial Basis Functions Based Level-Set Interpolation with an Ellipse Constraint

Similar to the recent works by Xie et al.² and Gelas et al.⁸, we interpolate the level-set function $\phi(\mathbf{x})$ using N RBFs in terms of $\mathbf{x}_i, i = 1, 2, \dots, N$. Each RBF function θ_i is a radially symmetric function centered at position $\mathbf{x}_i$. The RBF-based level-set function is expressed as

$$\phi_{RBF}(\mathbf{x}) = P(\mathbf{x}) + \sum_{i=1}^{i=N} \lambda_i \theta_i(\mathbf{x}) \quad (6)$$

where $\theta_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\|^2 \ln(\|\mathbf{x} - \mathbf{x}_i\| + 1)$. $\phi_{RBF}(\mathbf{x})$ can be considered as the blending of two implicit contours, one of which is modeled by the polynomial part, and the other by radial basis functions. Similar to⁹, we choose

$$P(\mathbf{x}) = ax^2 + bxy + cy^2 + dx + ey + f \quad (7)$$

The condition for constrainting $P(\mathbf{x})$ as an implicit form of an ellipse is $4ac - b^2 > 0$.

The RBF interpolants then can be obtained by solving the following linear system of equations:

$$\begin{bmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \Phi \quad (8)$$

where $\lambda_1 = [\lambda_1 \dots \lambda_N]^T$, $\lambda_2 = [a \ b \ c \ d \ e \ f]^T$,

$$\Phi = [\phi_{RBF}(\mathbf{x}_1) \dots \phi_{RBF}(\mathbf{x}_N) \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

and

$$\mathbf{P} = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N^2 & x_N y_N & y_N^2 & x_N & y_N & 1 \end{bmatrix} \quad \mathbf{A}_{i,j} = \varphi_j(\mathbf{x}_i), \quad i, j = 1, \dots, N$$

λ_2 can be obtained by solving the following QCQP problem.

$$\begin{cases} \text{Minimize} & \|\mathbf{D}\lambda_2 - \mathbf{c}\| \\ \text{Subject to} & \lambda_2^T \mathbf{E} \lambda_2 > 0 \quad (\Leftrightarrow 4ac - b^2 > 0) \end{cases} \quad (9)$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_0 & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \quad \mathbf{E}_0 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

where $\mathbf{D} = \mathbf{P}^T \mathbf{A}^{-1} \mathbf{P}$ and $\mathbf{c} = \mathbf{P}^T \mathbf{A}^{-1} \Phi$. As $\mathbf{E}$ is not a positive definite matrix, the above non-convex QCQP cannot be solved using the conventional techniques available in the field of nonlinear programming. Fortunately, we can relax it to a semi-definite programming (SDP), which can be solved easier. First of all, we homogenize the above problem as

$$\begin{cases} \text{Minimize} & \begin{bmatrix} \lambda_2^T & u \end{bmatrix} \begin{bmatrix} \mathbf{D}^T \mathbf{D} & -\mathbf{c}^T \mathbf{D} \\ -\mathbf{D}^T \mathbf{c} & \mathbf{c}^T \mathbf{c} \end{bmatrix} \begin{bmatrix} \lambda_2 \\ u \end{bmatrix} \\ \text{Subject to} & \begin{bmatrix} \lambda_2^T & u \end{bmatrix} \begin{bmatrix} \mathbf{E} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ u \end{bmatrix} > 0, \quad u^2 = 1 \end{cases} \quad (10)$$

Denote $\mathbf{X} = \begin{bmatrix} \lambda_2^T & u \end{bmatrix}^T \begin{bmatrix} \lambda_2^T & u \end{bmatrix}$. The non-convex QCQP has its relaxation as

$$\begin{cases} \text{Minimize} & \text{Trace} \left(\begin{bmatrix} \mathbf{D}^T \mathbf{D} & -\mathbf{c}^T \mathbf{D} \\ -\mathbf{D}^T \mathbf{c} & \mathbf{c}^T \mathbf{c} \end{bmatrix} \mathbf{X} \right) \\ \text{Subject to} & \text{Trace} (\mathbf{E} \mathbf{X}) > 0 \\ & \text{Trace} \left(\begin{bmatrix} \mathbf{0}_{6 \times 6} & \mathbf{0}_{1 \times 6} \\ \mathbf{0}_{6 \times 1} & 1_{1 \times 1} \end{bmatrix} \mathbf{X} \right) = 1, \quad \text{rank} (\mathbf{X}) = 1 \end{cases} \quad (11)$$

The constraint $\text{rank}(\mathbf{X})=1$ is non-convex but we can just skip it because according to¹⁰, the above SDP with only two constraints has one unique global optimal rank-one solution. We use SDPT3 solver¹² to find the solution of the above SDP. If the solution has the rank of one, it is also the solution of the QCQP. If the solution has a rank higher than one, we use the method in¹³ to extract the rank-one solution from that solution.

2.3 Active Modeling Using RBF Level-Set with Ellipse Constraint

The deformation of our deformable model is achieved by propagating the RBFs based level-set $\phi_{RBF}(\mathbf{x}, t)$ according to a speed function which is image dependent. It can be expressed as the following PDE:

$$\frac{\partial \phi_{RBF}(\mathbf{x}, t)}{\partial t} = F'(\mathbf{x}, t) |\nabla \phi_{RBF}(\mathbf{x}, t)| \quad (12)$$

where F' is the region-based speed function based on the Chan–Vese model, which aims at partitioning the image into regions with piecewise-constant intensity. The regularization term is no longer necessary as the smoothness of the contour is guaranteed by the RBFs based level-set representation.

$$F'(\mathbf{x}, t) = (I(\mathbf{x}) - \mu_{in})^2 - (I(\mathbf{x}) - \mu_{out})^2 \quad (13)$$

Modifying the speed function $F'(\mathbf{x}, t)$ by dividing it for $|\nabla \phi(\mathbf{x}, t)|$ to avoid flattening effects², the evolution equation becomes

$$\frac{\partial \phi_{RBF}(\mathbf{x}, t)}{\partial t} = F'(\mathbf{x}, t) \quad (14)$$

Following Wang et. al⁶, we assume that time and space are separable and the time dependence of the RBF based level-set function is now due to the RBF expansion coefficients λ .

$$\phi_{RBF}(\mathbf{x}, t) = a(t)x^2 + b(t)xy + c(t)y^2 + d(t)x + e(t)y + f(t) + \sum_{i=1}^{i=N} \lambda_i(t) \theta_i(\mathbf{x}) \quad (15)$$

Updating the level-set function is now considered as updating the RBF expansion coefficients. From (14) and combining with the RBFs based level-set interpolation with an ellipse constraint in section 2.2, the evolution process is given in the matrix form as

$$\left\{ \begin{array}{l} \Phi^t = \begin{bmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \lambda_1^t \\ \lambda_2^t \end{bmatrix} \\ \overline{\Phi} = \Phi^t - \mu \mathbf{B}^t \\ \text{Finding } \lambda_2^{t+1} \text{ from } \left\{ \begin{array}{l} \text{Minimize } \|\mathbf{D}^t \lambda_2^{t+1} - \mathbf{c}^t\| \\ \mathbf{D} = \mathbf{P}^T \mathbf{A}^{-1} \mathbf{P}, \quad \mathbf{c}^t = \mathbf{P}^T \mathbf{A}^{-1} \overline{\Phi} \\ \text{Subject to } (\lambda_2^{t+1})^T \mathbf{E} \lambda_2^{t+1} > 0 \end{array} \right. \\ \lambda_1^{t+1} = \mathbf{A}^{-1} (\overline{\Phi} - \mathbf{P} \lambda_2^{t+1}) \end{array} \right. \quad (16)$$

where

$$\mathbf{B} = \begin{bmatrix} F'(\mathbf{x}_1, t) \\ F'(\mathbf{x}_2, t) \\ \dots \\ F'(\mathbf{x}_N, t) \end{bmatrix}$$

The non-convex QCQP is relaxed to the SDP and solved following the procedure in section 1.2.

3. RESULTS

The proposed deformable model was simulated in a MATLAB 7 environment using a Pentium Dual Core, 2.0 GHz processor with 4 GB of memory, and under the Microsoft Windows 7 operating system. Two well-known models, Chan-Vese and Bernard, were chosen for comparison with our method. Fig. 1 demonstrates the performance of our deformable model in segmenting images containing complex topological shapes with a sparse mesh grid for the RBF centers. The mesh grid for our model is one RBF center placed in every 4x4 pixels domain. Chan-Vese and Bernard models were also used to segment these objects. The knots of the B-spline of the Bernard model were located on the same mesh grid with our model (regular spacing $h = 4$). Only the proposed model can capture all the objects exactly with this sparse mesh grid. Fig. 2 shows the performance of three models in segmenting a noisy hand-written Chinese character of size 275x400 with the same mesh grid of the experiments in Fig. 1. For all experiments in Figs. 1 and 2, our model converged within 10 iterations.

Figure 1. From left column to right column: Initialization, Chan-Vese model, Bernard model, our proposed model.

Figure 2. From left column to right column: Initialization, Chan-Vese model, Bernard model, our proposed model.

4. CONCLUSIONS

We have presented a new deformable model capable of segmenting images containing complex topological shapes using a novel RBFs based level-set interpolation with an ellipse constraint. In the present method, the object image extraction problem has been converted to the successive solution of an optimization problem in an iterative manner. An effective and efficient technique has been proposed to solve the optimization problem. The proposed model can extract the object images of complex shapes with severe intensity inhomogeneity using a sparse mesh grid for the RBF centers. The proposed model can thus segment exactly very complex images with a sparse mesh grid for the RBF centers. In addition, the comparison with other

methods has showed that our model is relatively insensitive to the location of initialization.

5. REFERENCES

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